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Abstract

This paper develops a dynamic general equilibrium model in infinite horizon, with an endogenous banking sector, market sensitive regulatory constraints, and in which deterministic rational housing bubbles may emerge. We are interested in the conditions under which housing bubbles may emerge and their impact on the economy. We show that 1) when agents face a LTV regulation, two different equilibria may emerge and coexist: a bubbleless and a housing bubble equilibria; 2) housing bubbles increase banks' size; 3) when banks face operational costs, housing bubbles reduce welfare. In an extension of the model we introduce a stochastic banking bubble and show that the combination of two market sensitive macroprudential regulations, LTV and VaR regulations, allows housing and banking bubbles to arise simultaneously. Their interaction amplifies banks' balance sheet size. The welfare impact is positive.

JEL classification: E44; E60; G1; G21; G21

Key words: Banking bubble; Dynamic general equilibrium; Housing bubbles; Infinitely lived agents; Loan-to-Value; Market sensitive regulations.

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1 Introduction

The Great Recession of 2007-2009 has highlighted the importance of housing bubbles in the build up of vulnerabilities in the economy as a whole and, in particular for the banking sector. Banking market values were tied to house prices and strongly decreased with the collapse of the U.S. housing bubble. The banking sector then experienced large losses, leading to a worldwide crisis. The aftermath of the U.S. housing bubble burst gave policy makers and macroeconomists a challenge.

Bearing in mind the consequences of the last financial crisis, and observing a large increase in house prices in many European countries, authorities are building some pre-emptive tools to protect the banking sector from vulnerabilities arising in the real estate sector. These measures, often called *borrower based measures* or *non interest policies*, aim at protecting the banking system against the pitfall of having loose credit standards for house purchase. Loose credit standards increase bank vulnerabilities and therefore increase systemic risk. In that context, macroprudential authorities across the world have started implementing *borrower based measures*. For instance, Belgium, Estonia, Luxembourg, the Netherlands and many other Euro area countries have recently introduced a *borrower based measure* called the Loan-to-Value (LTV) regulation (see ESRB and national notifications).[1] LTVs are market sensitive lending constraints: the total amount of credit that can be granted to a borrower for the purchase of a property cannot exceed a fraction of the property value.

While the aim of market sensitive regulations was to increase bank sensitivity to market risk, regulators admit it may have opened a Pandora's box. During the EBA Policy Research Workshop, 12 November 2020 introductory speech, Carolyn Rogers, Secretary General of the Basel Committee on Banking supervision, highlighted that the Value-at-Risk (VaR) capital requirement,[2] a Basel regulation that imposes a solvency condition on banks where the maximum amount of debt that banks can hold is limited by the market value of banks' assets in the worst case

[1] See esrb.europa.eu/national_policy/other/html/index.en.html

[2] This regulation, introduced by the Basel Committee in the Basel II framework, imposes a solvency condition on banks

scenario, led to an underestimation of risk.[3] Moreover, Danıelsson et al. (2004) show that market sensitive regulations may lead to greater price volatility.

The objective of this study is to investigate the role of LTV and VaR regulations on the emergence of housing bubbles, and the impact on the economy. In an extension, we study the economic consequences of the interaction of housing and banking bubbles. The paper provides policy implications on LTV and VaR regulations, as well as tax policies.

We develop a dynamic general equilibrium model in infinite horizon, with four types of infinitely lived agents, banks, impatient and patient households, firms. In addition, there is a government which also plays the role of a regulatory authority. Patient households, consume, save and are shareholders of banks while impatient households consume, work and borrow from banks to invest in housing. Impatient households are constrained by a Loan-to-Value regulation that limits their ability to borrow by a fraction of the house value. In order to focus on the non-fundamental part of house prices, we assume that houses do not provide any rent nor utility. Houses can be therefore interpreted as any other type of assets. Extending the model to account for features that provide housing services, such as in Iacoviello (2005) is left for future research. Firms produce the consumption good using capital and labor, and borrow money from the bank. Banks raise funds using net worth, and by taking deposits supplied by patient households. Using raised funds, they lend to firms and to impatient households. Banks face capital requirements based on VaR as recommended by the Basel II and III accords. The government provides a mortgage interest deduction to house owners, the impatient households, and balance its budget by taxing dividend earners, the patient households. In this model, we introduce a housing bubble, defined as a positive deviation of the house price from its fundamental value. Housing bubbles emerge only if agents believe that the house price contains a bubble. Housing bubbles are, thus, self-fulfilling. Since houses do not provide any rent nor utility, if agents do not believe that a bubble exists, they will not be traded or used as collateral. In the extension of the model we borrow the framework of Chevallier and El Joueidi (2019) to introduce banking bubbles on banks stock prices. Banking bubbles, like housing bubbles, only emerge if agents believe they exist. In addition, banking

[3]EBA Policy Research Workshop, 12 November 2020.

bubbles arise with a probability of burst, i.e. they are stochastic.

Our main result says that LTV regulation allows housing bubbles to emerge. When agents face such regulation, two different equilibria can coexist: a *bubbleless* and a *housing bubble equilibria*. While housing bubbles increase banks' balance sheet size, they decrease welfare. We also show that the combination of two market sensitive macroprudential regulations, LTV and VaR regulations, allow housing and banking bubbles to arise and exist simultaneously. Banking bubbles amplifies the increase of banks' balance sheet size already triggered by the housing bubble. The welfare impact until the banking bubble bursts is positive.

This study is related to the literature on rational bubbles in infinite horizon. Tirole (1982) and Blanchard and Watson (1982) argue that bubbles under rational expectations do not satisfy the transversality condition (TVC) and thus cannot exist in infinitely lived agents models. The only reason to hold an asset on which there is a bubble is "to resell it at some time and to realise the expected capital gain. But if all agents intend to sell in finite time, nobody will be holding the asset thereafter, and this cannot be an equilibrium" (Blanchard and Watson, 1982). This would imply that agents over-save and thus hold positive wealth at infinity. However, Kocherlakota (1992), Santos and Woodford (1997), and Miao and Wang (2015) show that bubbles in infinitely lived agents models may exist when agents' portfolios are constrained. Indeed, Chevallier and El Joueidi (2019) show by means of an infinitely dynamic general equilibrium model with an endogenous banking sector that the VaR regulation permits rational banking bubbles to exist. The present paper focuses on housing regulation and the existence of housing bubbles. It contributes to the literature by showing that LTV regulation allows housing bubbles to emerge. We show that under LTV constraints, housing bubbles may satisfy the TVC. Our result is consistent with Miao et al. (2014). They develop an infinitely lived general equilibrium model, where entrepreneurs purchase housing and face idiosyncratic investment tax distortions and credit constraints. They demonstrate that when house trading is liquid, *borrower based measures* do not prevent housing bubbles from existing.

This paper also contributes to the literature on the effect of LTV regulations on the economy. We provide evidence that by tying asset market values to lending, LTV contributes to the emergence of housing bubbles. The empirical literature has provided some evidence that LTV reduces credit growth but has failed to provide evidence of its impact on house price growth (Cerutti et al. (2017), Kuttner and Shim (2016), and Vandenbussche et al. (2015)). Kuttner and Shim (2016) show that only tax changes have a significant impact on house prices. We contribute to this strand of literature by showing the conditions under which housing bubbles can appear.

Finally, we contribute to the literature by studying the conditions under which housing and banking bubbles exist simultaneously and how their interaction impacts the economy. To our knowledge, this is the first time such question is being investigated.

The paper is organised as follows. **Section 2** presents the Baseline model. **Section 3** analyses the general equilibrium model. **Section 4** extends the Baseline model by including banking bubbles. Finally, **Section 5** concludes.

2 Baseline model

Consider a deterministic economy with four types of infinitely lived agents, banks, impatient and patient households, and firms. There is, in addition, a government which plays the role of a regulatory authority. Time is discrete and indexed by the variable $t = 0, 1...$ Banks, patient and impatient households, and firms are respectively represented by a continuum of homogeneous agents of mass one. Patient households consume, save and are shareholders of banks. Impatient households derive utility from consumption, work and borrow from the bank to invest in housing. Household value housing. This is modelled by assuming they want to take an infinite amount of mortgage loans to purchase houses. It is also assumed housing price is too high to be bought or shortened using savings such that only impatient households hold housing. We consider the fixity of the housing stock in the aggregate. Banks have the technology to engage in lending activity. They raise funds using net worth and deposits supplied by patient households, and lend to firms and to impatient households. Firms use capital and labor to produce the consumption good. The government provides a mortgage interest deduction to impatient households and balances its budget by taxing dividend earners, the patient households.[4]

In our model, we introduce housing bubbles that cannot be ruled out by the transversality condition. Housing bubbles are positive deviations of house prices from their fundamentals. They emerge only if agents believe that the house price contains a bubble. They are, thus, self-fulfilling.

2.1 Patient households

Patient households are represented by a continuum of homogeneous agents of unit mass. Each household starts with an initial endowment of stocks s_0 and deposits D_0. At each period t, the representative household pays a lump sum tax T_t to the government, receives net profits π_t generated by firms, dividends d_t from the shares s_t it owns, sells its shares at price p_t, obtains an interest rate r_t on the amount deposited D_t in the previous period, and chooses its optimal consumption c_t, amount of stocks s_{t+1}, and deposits D_{t+1} for the next period. There is no uncertainty on savings and thus r_t is the risk-free interest rate. To simplify calculations we assume that preferences of households are represented by a linear utility function in consumption. Given their budget constraint (1), each household chooses the optimal amount of shares, deposits and consumption $\{s_{t+1}, D_{t+1}, c_t\}_{t=0}^{\infty}$ that maximises its lifetime linear utility. Each household optimisation problem is defined as follows

$$Max_{\{s_{t+1},D_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t,$$

subject to

$$D_t (1 + r_t) + s_t (p_t + d_t) + \pi_t = D_{t+1} + c_t + s_{t+1} p_t + T_t, \tag{1}$$

$$c_t,\ D_t,\ s_{t+1} \geq 0 \ \text{ for all } t,$$

where $\beta \in]0, 1[$ is the household discount factor.

The first order conditions with respect to D_{t+1} and s_{t+1}, are given by

[4] Banks, firms, patient and impatient households are distinct agents. Nevertheless, we assume for simplicity that they discount the future at the same rate.

$$\beta\left(1+r_{t+1}\right)=1, \tag{2}$$

$$p_t=\beta\left(d_{t+1}+p_{t+1}\right). \tag{3}$$

The combination of (2) and (3) gives the patient household no arbitrage condition, $\left(d_{t+1}+p_{t+1}\right)/p_t=1+r_{t+1}$. It shows that in equilibrium, patient households hold both deposits and stocks to the point where the return on stocks equals the return on deposits.

Conditions (2) and (3) are not sufficient conditions to detect an optimal path for the problem above. To define the necessary and sufficient conditions of the optimization problem above, consider in addition the following transversality condition (TVC)

$$lim_{T\to\infty}\beta^T p_T s_T=0. \tag{4}$$

The TVC (4) is a boundary condition to ensure that the patient household does not hold positive wealth nor over-accumulates stocks as the planning horizon tends towards infinity.

2.2 Firms

Firms are represented by a continuum of homogeneous producers of unit mass. Each firm starts with an amount of loans L_0, named corporate loans, to buy its initial capital K_0. In each period t, firms produce y_t using capital K_t and labor l_t bought in t. They reimburse their loans with interests i_t such that the total reimbursement is $L_t\left(1+i_t\right)$. The parameter A is the total factor productivity. Firms then distribute net profits to patient households and choose their optimal amount of labor, corporate loans and capital for the next period $\left\{l_t, L_{t+1}, K_{t+1}\right\}_{t=0}^{\infty}$ to maximise their future discounted profits π_t subject to their budget constraint (5), and the investment constraint (7). To simplify the model, we consider capital that fully depreciates such that the new stock of capital K_{t+1} is equal to the investment I_t, which is fully financed by taking corporate loans L_{t+1}. Each firm optimisation problem is defined as follows

$$Max_{\{L_{t+1},K_{t+1},l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \pi_t,$$

subject to

$$\pi_t = y_t - I_t + K_{t+1} - K_t (1 + i_t) - w_t l_t, \tag{5}$$

$$y_t = A{K_t}^{\psi} l_t^{1-\psi}, \tag{6}$$

$$K_{t+1} = I_t, \tag{7}$$

$$I_t = L_{t+1}, \tag{8}$$

$$\pi_t \geq 0 \text{ and } L_t,\ K_t > 0 \ \text{ for all } t,$$

where $\psi \in]0,1[$ is the output elasticity of capital. Using the Lagrange method, the interior solution of the first order condition with respect to L_{t+1} and l_t are given by

$$\psi A \left(\frac{K_{t+1}}{l_{t+1}} \right)^{\psi - 1} = 1 + i_{t+1}, \tag{9}$$

$$(1 - \psi) A \left(\frac{K_t}{l_t} \right)^{\psi} = w_t. \tag{10}$$

Equation (9) shows that, in the optimum, the marginal product of capital is equal to the marginal cost of corporate loans. Condition (10) implies that labor will be employed up to the point where the marginal product of labor is equal to the marginal cost of labor, w_t.

2.3 Banks

The banking sector is represented by a continuum of homogeneous banks of unit mass. Banks collect deposits D_{t+1} from patient households, accumulate net worth N_{t+1}, grant corporate loans L_{t+1} to the firms and mortgage loans M_{t+1} to impatient households. As required by central banks, banks are subject to reserve requirements,[5]

[5] These reserves are there to meet large unexpected withdrawals. Although such risk is not modelled here, these constraints potentially impact the existence of bubbles, which we study here.

$$R_t \equiv \phi D_t, \tag{11}$$

where $\phi \in [0,1[$ is the fraction of deposits that is kept as reserves. The bank balance sheet is

$$R_t + L_t + M_t = N_t + D_t. \tag{12}$$

Following Jensen (1986), and as modelled in Estrella (2004), Blum (2008) and Chevallier and El Joueidi (2019), we assume an agency problem between shareholders and managers of the bank when their interests diverge, making debt less expensive than net worth. There is a cost of holding net worth, $\mathbb{C}_t = \tau N_t$, where the parameter $\tau \in]0,1]$ is the opportunity cost of net worth. Let i_t be the lending rate earned on corporate loans in t, ρ_t the lending rate earned on mortgage loans in t, and r_t the risk-free interest rate paid in t. At the end of each period t, banks accumulate net worth using profits from loans granted in t, net of deposit repayments, dividends and the cost of net worth,

$$N_{t+1} = (1 + i_t) L_t + (1 + \rho_t) M_t + R_t - D_t (1 + r_t) - d_t - \mathbb{C}_t. \tag{13}$$

To avoid bank insolvency, Basel II and III recommend banks to hold a minimum level of capital by imposing them a Value-at-Risk constraint. As in Chevallier and El Joueidi (2019), banks face a Value-at-Risk regulation,[6]

$$D_t \leq \eta V_t (N_t), \tag{14}$$

where $\eta > 0$ is the Value-at-Risk regulation parameter. The Basel framework objective is to preserve a safety buffer for the banks to be able to repay depositors. The maximum amount of deposits banks can hold cannot exceed the market value of banks' assets in the worst case scenario. It is considered that banks want to hold as much deposits as they can since the marginal benefit from holding deposits is larger than the marginal cost of holding them. **Appendix A** shows that this consideration is equivalent to the following restriction on parameters

[6] See the BIS publication, the First Pillar Minimum Capital Requirements, http://www.bis.org/publ/bcbs107.html

$$\tau\beta\left(1-\phi\right) > \phi\left(1-\beta\right). \qquad (15)$$

Hence, the VaR capital regulation (14) always binds such that

$$D_t = \eta V_t\left(N_t\right). \qquad (16)$$

At the end of period t, each bank chooses its optimal net worth to accumulate for next period N_{t+1} and corporate loans M_{t+1} in order to maximise its current dividends and the present value of future dividends subject to reserve requirements (11), its balance sheet constraint (12), its budget constraint (13) and the VaR regulation (16). The value of the bank in period t is denoted $V_t\left(N_t\right)$.

The bank problem can be summarised by the following Bellman equation

$$V_t\left(N_t\right) = Max_{\{N_{t+1},M_{t+1}\}}\left\{d_t + \beta V_{t+1}\left(N_{t+1}\right)\right\}, \qquad (17)$$

subject to

$$d_t = \left(1+i_t\right)N_t + D_t\left[i_t(1-\phi) - r_t\right] + M_t\left(\rho_t - i_t\right) - \tau N_t - N_{t+1}, \qquad (18)$$

$$D_t = \eta V_t\left(N_t\right), \qquad (19)$$

$$N_{t+1},\ D_t,\ M_t \geq 0 \text{ for all } t.$$

Appendix B shows that the solution of the above maximisation problem gives the value function below

$$V_t\left(N_t\right) = q_t N_t, \qquad (20)$$

where $q_t \geq 0$ is the marginal value of net worth. This variable q_t may measure banks' performance and can also be interpreted as the banks' Tobin's Q ratio (Tobin, 1969).

The bank stock price in t is defined by

$$p_t = \beta V_{t+1}\left(N_{t+1}\right).$$

The system of equations describing the solution of each bank maximisation problem is derived in **Appendix B** and is summarized by the following equations

$$q_{t+1} = \frac{1}{\beta}, \tag{21}$$

$$q_t = (1 + i_t - \tau) + \eta q_t \left[i_t (1 - \phi) - r_t \right], \tag{22}$$

$$1 + \rho_t = 1 + i_t. \tag{23}$$

The constant value of the bank's Tobin's Q in equation (21) reflects risk neutrality. By raising one unit of net worth at time t, the bank gets the discounted marginal value of net worth. Equation (22) shows that an additional unit of net worth allows the bank to relax the constraint by taking η units of additional deposits, giving an additional return of $\eta \left[i_t (1 - \phi) - r_t \right]$. Consistent with the risk neutrality assumption, (21) and (22) show that the lending rate on corporate loans, i_t, is constant. Equation (23) is the no arbitrage condition on loans which shows that the return on mortgage loans should equal the one on corporate loans.

2.4 Impatient Households

Impatient households are represented by a continuum of homogeneous agents of unit mass. Each impatient household starts with an initial endowment of housing h_0 and mortgage loans M_0. At each period t, the representative impatient household chooses her optimal amount of debt M_{t+1} in order to buy a quantity h_{t+1} of houses. She also pays an interest rate ρ_t on the amount loans M_t she took in the previous period and decides her optimal consumption $c_{h,t}$ and labor supplied l_t. Labor provides her a wage w_t at period t. The regulator imposes a Loan-to-Value regulation: the maximum amount of mortgage loan that households can borrow is proportional to the value of the house purchased, $p_{h,t+1} h_{t+1}$, where $p_{h,t+1}$ is the house price in $t + 1$. The preferences of households are represented by a positive utility function in consumption and negative in labor.

In our model, beliefs are dual, agents believe or not housing bubbles can exist. Moreover, for simplicity and in order to focus on housing bubbles we assume that households do not derive any rent nor direct utility from housing. Hence, if agents do not believe that a housing bubble exists, houses will not be traded or used as collateral i.e., $p_{h,t} = 0$ for all t.

Extending the model to include utility or direct rent from housing would be an interesting path to follow and is left for future research. If agents believe a housing bubble exists, we will show that an equilibrium with a housing bubble, $p_{h,t}>0$ for all t may emerge.

At the end of each period t, each impatient household chooses the optimal amount of loans, work, houses and consumption $\{M_{t+1}, l_t, h_{t+1}, c_{h,t}\}_{t=0}^{\infty}$ subject to their budget constraint (25), and the Loan-to-Value constraint (26), to maximise its lifetime linear utility. Each impatient household optimisation problem is defined as follows

$$Max_{\{M_{t+1},l_t,h_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[c_{h,t} - n\frac{l_t^{1+\varphi}}{1+\varphi}\right], \tag{24}$$

subject to three conditions

$$c_{h,t} = M_{t+1} - [1 + \rho_t - \sigma] M_t - p_{h,t}(h_{t+1} - h_t) + w_t l_t, \tag{25}$$

$$M_{t+1} \leq m p_{h,t+1} h_{t+1}, \tag{26}$$

$$h_t,\ M_t,\ l_t,\ c_{h,t} \geq 0 \quad \text{for all } t, \tag{27}$$

where $\beta \in]0, 1[$ is the household discount factor, $\sigma > 0$ is a tax advantage from borrowing, and $m > 0$ is the regulatory maximum LTV ratio. This tax advantage takes the form of a mortgage interest deduction, which is a popular political tax incentive (Morrow, 2012). The parameter φ is the inverse of the Frish elasticity of labor supply. It captures by how much labor supply responds to a change in the wage. Its value ranges between 0.1 and 10 (when 10, labor supply is inelastic). It will later be calibrated to the model.

It is assumed that one unit of housing gives more than its discounted marginal cost so that households always desire to hold housing. **Appendix C** shows that, in our model, it is equivalent to the restriction $1/\beta > 1+\rho_t-\sigma$, which corresponds to the following restriction on parameters

$$\sigma > \frac{r\eta\phi + \beta\tau}{\beta + \eta(1-\phi)}. \tag{28}$$

Hence, the Loan-to-Value constraint (26) always binds and becomes

$$M_{t+1} = mp_{h,t+1}h_{t+1}. \tag{29}$$

Housing bubbleless path

When agents do not believe that a housing bubble exists in period t, given that agents do not value housing in their utility nor derive rent from it, house prices are null, i.e. $p_{h,t} = 0$.

Considering the impatient household problem above, the necessary and sufficient conditions for an optimal housing demand h_{t+1}, mortgage loans demand M_{t+1}, and optimal labor supply l_t, are $h_{t+1} = 0$, $M_{t+1} = 0$, and $w_t = nl_t^{\varphi}$.

Housing bubbly path

When agents believe that a housing bubble exists in period t, house prices contain a bubble, i.e. $p_{h,t} > 0$.

Considering the impatient household problem above with the binding Loan-to-Value constraint (29), the necessary and sufficient conditions for an optimal housing demand h_{t+1} and optimal labor supply l_t are the interior solutions (30) and (31), together with the terminal condition (32) below,

$$m + \beta - \beta m \left(1 + \rho_{t+1} - \sigma\right) = \frac{p_{h,t}}{p_{h,t+1}}, \tag{30}$$

$$w_t = nl_t^{\varphi}, \tag{31}$$

$$lim_{T \to \infty} \beta^T p_{h,T} h_T = 0. \tag{32}$$

Equation (30) imposes equality between the marginal rates of substitution between housing today and tomorrow and the marginal product of housing. Equation (31) equates the wage with the marginal utility of labor. The terminal condition (32) is the transversality condition associated with housing. Together with (30) and (31), this second TVC characterizes the optimal path of the model. It ensures that there are no unused resources at the end of the planning period. Tirole (1982) shows that bubbles under rational expectations with infinitely lived agents cannot exist since the TVC cannot be satisfied. In our model, we will show that housing bubbles may satisfy the housing TVC and thus, may exist.

Proposition 1. *If*

$$\frac{1}{\beta} > 1 + \rho_{t+1} - \sigma, \tag{33}$$

the transversality condition of the impatient household (32) is always satisfied and therefore, a housing bubble may appear.

The proof is presented in **Appendix D**. This Proposition 1 shows that the TVC (32) is satisfied, i.e. bubbles are not ruled out, if the marginal cost of borrowing is not larger than the rate of time preference of households. By the bank optimal solutions (22), (22), and (23), $\rho_{t+1} = \rho$, and (33) is equivalent our initial assumption on the model, the condition of a binding LTV regulation (cf. equation (28)). Hence, if agents believe a housing bubble exists, and impatient households are willing to take an infinite amount of mortgage loans, condition (33) is always satisfied and a housing bubble may emerge, i.e. $p_{h,t} > 0$.

Proposition 2. *Loan-to-Value regulation allows housing bubbles to exist.*

The proof of Proposition 2 is the following. Rearranging the intertemporal condition of housing (30), the growth rate of house prices becomes

$$\frac{p_{h,t+1}}{p_{h,t}} = \frac{1}{\beta + m\left\{1 - \beta\left[1 + \rho_{t+1} - \sigma\right]\right\}}. \tag{34}$$

The above equation shows that without the Loan-to-Value regulation (equivalent to $m = 0$), the growth rate of house prices is $p_{h,t+1}/p_{h,t} = 1/\beta$, which is ruled out by the housing TVC (32). If it doesn't satisfy the TVC, it means gains can be achieved by deviating from the housing bubbly path. Thus, an equilibrium with a a housing bubble cannot exist. In contrast, when the impatient household faces a constraint on lending that is tied to the value of the house ($m > 0$), the growth rate of house prices is reduced (since it is smaller than the discount rate $1/\beta$), allowing bubbles to emerge.[7]

[7] Note that $\beta\left[1 + \rho_{t+1} - \sigma\right] < 1$ by condition (28).

The intuition is that without LTV regulation, people borrow as much as possible to buy housing, increasing housing demand and then housing price growth. When agents believe a bubble exists, housing prices grow too fast such that an explosive bubble might appear. However, as explained by Tirole (1982), an equilibrium with an explosive housing bubble cannot emerge since "the only reason to hold an asset whose price is above its fundamental value is to resell it at some time and to realise the expected capital gain. But if all agents intend to sell in finite time, nobody will be holding the asset thereafter, and this cannot be an equilibrium". By constraining agents' ability to borrow and thus purchase housing, the LTV reduces the housing price growth rate, and thus, a non-explosive bubble can satisfy the TVC. This result, captured by Proposition 2, is at the center of our paper. It shows that the recent policy trend of imposing LTV limits may not be the appropriate tool to curb housing booms. Proposition 2 is also consistent with the result of Danıelsson et al. (2004) who show that price dynamics are exacerbated by regulations that are tied to market values.

Another intuition of why the housing TVC is satisfied with a housing bubble is captured by writing condition (34) as

$$\underbrace{p_{h,t}\left(\frac{1}{\beta}-1\right)}_{\text{housing return}} = \underbrace{p_{h,t+1}-p_{h,t}}_{\text{capital gain}} + \underbrace{p_{h,t}\left(\frac{1}{\beta}-\frac{1}{\beta+m\left\{1-\beta\left[1+\rho_{t+1}-\sigma\right]\right\}}\right)}_{\text{dividend yield}}. \quad (35)$$

The above equation shows that the return on housing is equal to a capital gain $p_{h,t+1}-p_{h,t}$ plus a dividend yield. In bubbleless models, this dividend yield term is inexistent. The dividend yield in the infinite horizon model guarantees that the transversality condition does not rule out the bubble. The intuition is that a liquidity premium is provided by the bubble. An additional unit of the bubble allows the impatient households to relax the Loan-to-Value constraint by m units, and hence to increase mortgage loans by m units. These additional units of loans allow impatient households to increase their purchase of houses. As long as there is a positive dividend yield term, the growth rate of the housing price is lower than the household rate of time preference. Hence, the TVC is satisfied.

Proposition 3. *When agents face an LTV, the mortgage interest deduction should* *be large enough to allow bubbles to exist.*

Proposition 3 follows from condition (33). This condition shows that a small enough interest rate on loans coupled with a high enough tax advantage on house purchase allow the terminal condition for an optimal path, i.e. the TVC, to be satisfied. Hence, the combination of this tax incentive and the borrower based regulation allow an equilibrium with a housing bubble to exists. The literature often finds that mortgage interest deductions policies, which are commonly used in advanced economies, contribute to inflate house prices (Morrow, 2012). Our Proposition 3 contributes to this strand of literature by showing that mortgage interest tax deductions, coupled with binding LTV limits, inflate house prices by allowing housing bubbles to emerge.

3 General equilibrium

This section presents two equilibria, the *bubbleless equilibrium*, defined by an equilibrium in which no bubble emerge, and an equilibrium with a housing bubble. It also presents the conditions under which these two equilibria may coexist. At equilibrium, the quantity of stock and total housing supply are normalized to one ($\forall t,\ s_{t+1} = 1,\ h_{t+1} = 1$).

3.1 Bubbleless

In this subsection, we present and analyse *the competitive general bubbleless equilibrium*. It is defined by an economy without bubbles ($\forall t,\ p_{h,t} = 0$). Variables are denoted x_t^*, $t \epsilon [0; \infty[$.

Definition 4. *A competitive general bubbleless equilibrium*, with $p_{h,t}^* = 0$ for all t, is defined as sequences of allocations and prices

$$\begin{aligned} \mathcal{E}_t^* \quad = \quad & \{d_t^*, N_{t+1}^*, K_{t+1}^*, L_{t+1}^*, M_{t+1}^*, D_{t+1}^*, \pi_t^*, y_t^*, \\ & c_t^*, s_{t+1}^*, q_t^*, r_t, i_t^*, \rho_t^*, p_t^*, c_{h,t}^*, l_t^*, w_t^*, h_{t+1}^*\} \ \ \forall t, \end{aligned}$$

such that taking prices as given, all agents maximise their future payoffs subject to their constraints. Transversality conditions (32) and (4) being satisfied. Also, the market for loans, for deposits, for housing ($h^*_{t+1} = 1$), and for stocks ($s^*_{t+1} = 1$) clear, and the government's budget is in equilibrium such that the lump sum tax T^*_t to patient households equals the government costs from the mortgage tax advantage provided to homebuyers, σM^*_t. From the three budget constraints (1), (5), and (13), aggregate equilibrium consumption is given by output and labor earnings net of investment, plus variation in reserves, minus the cost of net worth

$$c^*_t + c^*_{h,t} = y^*_t + w^*_t l^*_t - L^*_{t+1} - \left(R^*_{t+1} - R^*_t\right) - \tau N^*_t. \tag{36}$$

Stationary bubbleless equilibrium

A *stationary bubbleless equilibrium* is characterized by a constant path where variables over time are defined by $\mathcal{E}^*_0 = ... = \mathcal{E}^*_t = \mathcal{E}^*$ for all t. The equilibrium deposit rate given by (2) is $r = 1/\beta - 1$. The stationary marginal value of net worth given by (21) is $q^* = 1/\beta$. From the VaR regulation (16) and the value function (20), the stationary bank leverage, D^*/N^*, depends negatively on the discount factor, β, and positively on the lenience of the VaR regulation, η, according to

$$\frac{D^*}{N^*} = \frac{\eta}{\beta}. \tag{37}$$

The stationary lending rates are then given by the Euler equation (21) and (22), such that $i^* = \rho^*$, and

$$i^* = \frac{r\left(\eta + \beta\right) + \beta\tau}{\beta + \eta\left(1 - \phi\right)}. \tag{38}$$

Larger opportunity costs of net worth, τ, and reserves, ϕ, decrease the supply of loans, putting upward pressure on the lending rate i^*. In contrast, looser VaR regulation, i.e. large η, allows banks to demand more deposits, raising bank's size and loan supply, thereby reducing the lending rate.[8] Assuming no frictions nor

[8] It is easy to show that

$$\frac{\partial i^*}{\partial \eta} = \frac{(1-\beta) - \left[1 - \beta\left(1-\tau\right)\right]\left(1-\phi\right)}{\left[\beta + \eta\left(1-\phi\right)\right]^2} < 0.$$

The numerator is negative if and only if $\tau\beta(1-\phi) > \phi\left(1-\beta\right)$, which is always satisfied (cf. equation (15)).

costs in the banking sector, i.e. $\phi = \tau = 0$, the model collapses to a standard dynamic general equilibrium model with no bank.

The stationary impatient household wage is given by (10) and (9), such that $w^* = A\left(\left[\left(1+i^*\right)/\psi\right]^{1/(\psi-1)}\right)^{\psi}$. Hence, by the labor supply condition (31) we have $l^* = \left(w^*/n\right)^{1/\varphi}$. By the first order condition (9), the stationary level of corporate loans is

$$L^* \;=\; l^*\left[\left(1+i^*\right)/\psi\right]^{1/(\psi-1)}. \tag{39}$$

In the *stationary bubbleless equilibrium*, $p_h^* = 0$. As a consequence, from the LTV regulation (29), $M^* = 0$. From the balance sheet constraint (12), (37), and the LTV constraint (29), the equilibrium value of bank net worth is $N^* = \left(L^* + M^*\right)/\left[1 + \left(1-\phi\right)\left(\eta/\beta\right)\right]$, and from the VaR regulation (16), deposits are $D^* = N^*\eta/\beta$. From (18), $d^* = \left(i^* - \tau\right)N^* + D^*\left[i^*(1-\phi) - 1/\beta + 1\right]$. From the government budget equilibrium, $T^* = \sigma M^*$. The stationary equilibrium consumption for the patient household , $c^* = \left(i^* - \tau\right)N^* + D^* i^*(1-\phi)$, is deduced from (1) and (18). The impatient household consumption is derived from (25) and is thus only derived from labor earnings. Thus, $c_h^* = w^* l^*$. Finally, denote the *welfare in a stationary bubbleless equilibrium* as W^*. It is defined as

$$W^* = c^* + c_h^* \tag{40}$$

.

3.2 Housing bubble

In this section, the emergence and impact of housing bubbles on the economy is studied. We define and analyse the *housing bubble general equilibrium*, where house prices can be above zero although there is no rent nor utility derived from housing ownership. Denote x_t^h the variables in the general equilibrium in which housing bubbles exist.

Definition 5. A *competitive general equilibrium with a housing bubble*, such that

$p^h_{h,t} > 0$ for all t, is defined as sequences of allocations and prices

$$\begin{aligned}\mathcal{E}^h_t &= \{d^h_t, N^h_{t+1}, K^h_{t+1}, L^h_{t+1}, M^h_{t+1}, D^h_{t+1}, \pi^h_t, y^h_t, \\ &\quad c^h_t, c^h_{h,t}, s^h_{t+1}, q^h_t, r^h_t, i^h_t, p^h_{h,t}, h^h_{t+1}, l^h_t, w^h_t, T^h_t\} \;\; \forall t,\end{aligned}$$

such that taking prices as given, all agents maximise their future payoffs subject to their constraints. The transversality condition (32) being satisfied. Also, the market for loans clear, as well as the market for deposits, for housing , for labor, and for stocks. Moreover, the government's budget is in equilibrium, i.e. $\sigma M^h_t = T^h_t$. Aggregate equilibrium consumption is derived using the three budget constraints (1), (5), and (13). This aggregate equilibrium, also defined as the *housing bubble welfare* W^h_t, is expressed by the equality $c^h_t + c^h_{h,t} = y^h_t - L^h_{t+1} - \left(R^h_{t+1} - R^h_t\right) - \tau N^h_t + w^h_t l^h_t$.

Stationary housing bubble equilibrium

Here, we analyse the *stationary housing bubble equilibrium* such that variables are constant, i.e. $\mathcal{E}^h_0 = ... = \mathcal{E}^h_t = \mathcal{E}^h$ for all t. From (21), banks' marginal value of net worth, q^h, is given by $q^h = q^*$. From the bank no arbitrage condition on bank assets (23), $\rho^h = i^h$, where i^h is determined by the same conditions than the bubbleless interest rate ((22) and (21)). Thus

$$i^h = i^*. \tag{41}$$

Then, by the above condition, the firm's optimal conditions (9) and (10), the labor supply condition (31), (5) and (6), the following equalities hold: $w^h = w^*$, $l^h = l^*$, $K^h = K^*$, $\pi^h = \pi^*$, $y^h = y^*$, and

$$L^h = L^*. \tag{42}$$

In the *stationary housing bubble equilibrium*, house prices are characterized by $p^h_h > 0$. Mortgage loans are given by the Loan-to-Value constraint (29),

$$M^h = mp^h_h. \tag{43}$$

From the impatient household budget, the stationary equilibrium impatient household consumption is $c_h^h = M^h\left(\sigma - \rho^h\right) + w^h l^h$. The stationary level of dividends d^h is deduced from (18), $d^h = \left(i^h - \tau\right) N^h + D^h\left[i^h(1-\phi) - 1/\beta + 1\right]$. The patient household stationary equilibrium consumption is derived from (1), such that $c^h = D^h\left(1/\beta - 1\right) + d^h - T^h$, where $T^h = \sigma M^h$ (from the equilibrium of the government budget). From the constraints (12), (37), (26) and the VaR regulation in equilibrium, the bank's net worth and deposit value are given by $N^h = \left(L^h + M^h\right)/\left[1 + (1-\phi)\left(\eta/\beta\right)\right]$ and $D^h = N^h\eta/\beta$. Finally, the stationary housing bubble welfare is

$$W^h = c^h + c_h^h. \tag{44}$$

3.3 Equilibria comparison

This section compares the *housing bubble* and *bubbleless equilibria*, analyses the impact of both equilibria on the economy, and presents the conditions under which both equilibria may coexist.

Proposition 6. *A bubbleless and a housing bubble equilibria always coexist.*

Proposition 6 derives from the fact that, for the same set of parameters in our model, house prices can take two different values, $p_{h,t} = 0$ or $p_{h,t}>0$, where, $p_{h,t} = 0$ for all t characterizes the *bubbleless equilibrium*, and $p_{h,t}>0$ for all t the *housing bubble equilibrium.*

Proposition 7. *Banks' balance sheet is larger in the stationary housing bubble equilibrium than in the bubbleless stationary equilibrium.*

The proof is straightforward. Bank total assets are composed of corporate and mortgage loans. Equality (42) says that the level of corporate loans in the *stationary bubbleless* and *stationary housing bubble equilibria* are the same, i.e., $L^* = L^h$. Moreover, in contrast with the *stationary bubbleless equilibrium* where $M^* = 0$, the quantity of mortgage loans is positive when a housing bubble exists, i.e. $M^h > 0$. This implies that, keeping everything else constant, banks grant more loans when a bubble on housing exists, that is $M^h > M^*$. In particular, banks' exposure to real estate increases when a housing bubble exists. As a consequence, banks' assets, which is equal to the sum of net worth and deposits are also larger in the *housing bubble equilibrium*, i.e.

$$N^h = \left(L^h + M^h\right) / \left[1 + (1 - \phi)(\eta/\beta)\right] > N^*, \tag{45}$$

and

$$D^h = N^h \eta/\beta > D^*. \tag{46}$$

Proposition 8. *Given positive operational costs, i.e.* $\tau > 0$, *the welfare in a stationary housing bubble equilibrium is smaller than in a bubbleless stationary equilibrium.*

The proof of Proposition 8 is presented in **Appendix E**. From this appendix, it can be seen that if there are no operational costs $\tau = 0$, welfares in each of the two equilibria are equal. The intuition is the following. Operational costs reduce dividends, and thus the patient household consumption in both equilibria. However, the patient household dividends and thus consumption in the *stationary housing bubble equilibrium* are reduced even more because housing bubbles increase operational costs. Indeed, operational costs are proportional to banks' size, which increases with housing bubbles (see Proposition 7).

3.4 Equilibrium dynamics

This section analyses the local dynamics around the *stationary bubbleless* and *housing bubble equilibria*. A case of stability of the system is presented in the following numerical example.[9]

We calibrate the parameters and calculate the implied values of the variables in each equilibrium. The tightness of the Loan-to-Value constraint, m, is 1.1, meaning that agents can borrow at most 110% of the value of the house. The discount factor, the total factor productivity, and the capital share are calibrated

[9] All the parameter restrictions are listed in **Appendix I**.

to standard values in the literature such that $\beta = 0.99$, A=1, and $\psi = 0.33$. The VaR regulatory parameter is set to $\eta = 19$. This calibration allows to have a tier 1 ratio of about 7% in the stationary equilibrium with the housing bubble, slightly smaller than the 8% recommended by the Basel committee.[10] This ratio is 5% for the *stationary bubbleless equilibrium*. The reserve parameter $\phi = 0.01$ is calibrated to match the European Central Bank requirements.[11] Finally, we set the opportunity cost of net worth $\tau = 0.001$ so that operational costs represent 0.1% of banks' net worth at time t. Finally, the tax advantage on mortgage loans is set to 0.93%, i.e. $\sigma = 0.0093$, and the Frish elasticity of labor supply φ is calibrated to 0.1. This calibration satisfies (28), and thus, Propositions 1 and 6 show that, under these values, the *bubbleless* and the *housing bubble equilibria* coexist. Table 1 displays the implied steady state values.

Table 1: Steady states of the bubbly and bubbleless equilibria

	No bubble	Housing bubble only
	$(p_{h,t} = 0)$	$(p_{h,t} > 0)$
Variables	values	values
N	0.049638	0.073784
D	0.95264	1.41605
L	0.99276	0.99276
M	0	0.48293
$L + M$	0.99276	1.47568
p	0.049638	0.073784
p_h	0	0.439026
i	0.01	0.01
c	0.010124	0.010566
c_h	2.03616	2.03571
welfare	2.046284	2.046276

Compared to the bubbleless steady state, and as shown in Proposition 7, the quantity of loans supplied by banks, $L + M$, is larger in the *stationary housing bubble equilibrium*. Corporate loans L are the same in both equilibria. As pointed in Proposition 8, welfare is smaller under the *stationary housing bubble equilibrium*.

To analyse the stability and uniqueness properties of the system, we log-

[10]This ratio is defined as the total net worth over risky assets.

[11]See https://www.ecb.europa.eu/mopo/implement/mr/html/calc.en.html.

linearise the system around both stationary equilibria. This results in a system of linear difference equations. When a housing bubble does not exist, $p_{h,t} = 0$ for all t, as well as when a housing bubble exists, $p_{h,t} > 0$ for all t, the eigenvalues associated with the linearised system around, respectively, the *stationary bubbleless* and the *stationary housing bubble equilibria*, are such that the number of unstable eigenvalues (eigenvalues that lie outside the unit circle) is equal to the number of forward looking variables. All eigenvalues are reported in **Appendix F**. This implies that, under this calibration, the systems of equations when $p_{h,t} = 0$ and when $p_{h,t} > 0$ for all t, are determined, and both the *bubbleless* and the *housing bubble equilibria* are stable and unique. Thus, given an initial value of N_t^* in a neighbourhood of the *stationary bubbleless equilibrium*, there exists a unique value of $p_{h,t}^*$ such that the system of linear difference equations converges to the unique *stationary bubbleless equilibrium* along a unique saddle path (see Blanchard and Kahn, 1980). Similarly, given an initial value of N_t^h in the neighbourhood of the *stationary housing bubble equilibrium*, there exists a value of $p_{h,t}^h > 0$ such that the system of linear difference equations converges to a unique equilibrium with a housing bubble along a unique saddle path (Blanchard and Kahn, 1980).

4 Housing and banking bubbles

In this section, we introduce a banking bubble in the model based on the framework developed in Chevallier and El Joueidi (2019). We analyse the conditions under which housing and banking bubbles can coexist, their interaction and the impact of this coexistence on the economy. Housing bubbles are defined as in the Baseline model described in Section 2. A banking bubble is defined as a positive temporary deviation of the bank stock price p_t from its fundamental value. As already mentioned above, the existence of housing and banking bubbles depend on beliefs. They emerge only if agents believe that house prices and/or bank stock prices contain bubbles. The bubbles cannot be ruled out by the transversality condition*s.* As in Blanchard and Watson (1982), Miao and Wang (2015), and Chevallier and El Joueidi (2019), the banking bubble, denoted b_t, has an exogenous probability ξ of burst. We assume that once the banking bubble bursts in t, i.e. b_t=0, the housing bubble does not survive and the economy falls back to the bubbleless economy.[12] As observed during the 2008 financial crisis, when a banking bubble bursts, optimistic beliefs about housing is not sustainable since these two markets are intertwined.

We now describe the bank's problem with a banking bubble. The bank's problem remains the same as in the Baseline model (Section 2) except for the Bellman equation. The Bellman equation (17) in the Baseline model with a housing bubble (see Section 2) is replaced by

$$V_t\left(N_t\right) = Max_{\{N_{t+1},M_{t+1}\}}\left\{d_t + \beta\left[(1-\xi)V_{t+1}\left(N_{t+1}\right) + \xi\overline{V}_{t+1}\left(N_{t+1}\right)\right]\right\}. \tag{47}$$

When agents believe that a banking bubble exists in period t, i.e. $b_t > 0$, the bank's value in $t+1$ can take two different values: $V_{t+1}\left(N_{t+1}\right)$ with a probability $(1-\xi)$, $\xi \in]0,1[$, if the banking bubble survives, or $\overline{V}_{t+1}\left(N_{t+1}\right)$ with a probability ξ when the bubble bursts in $t+1$, i.e. $b_{t+1} = 0$. In the latter case, the model converges to the *bubbleless stationary equilibrium*. $\overline{V}_{t+1}\left(N_{t+1}\right)$ is defined as in the bank problem analysed in Subsection 2.3.

Following Chevallier and El Joueidi (2019), until the bubble's collapse, the solution of the bank maximisation problem gives the value function below

$$V_t\left(N_t\right) = q_t N_t + b_t. \tag{48}$$

where $b_t > 0$ and $q_t \geq 0$. The variable q_t is the marginal value of net worth when there is a banking bubble. Variables q_t and b_t are endogenously determined.

The bank stock price in t with a banking bubble, and before the bubble bursts, is

$$p_t = \beta\left[(1-\xi)V_{t+1}\left(N_{t+1}\right) + \xi\overline{V}_{t+1}\left(N_{t+1}\right)\right].$$

Until the bubble bursts, the solution of each bank maximisation problem is

[12] A housing bubble can reappear if there is a change in the parameter of the model.

given by

$$
\begin{aligned}
q_{t+1} &= \frac{1-\xi\beta\overline{q}_{t+1}}{\beta(1-\xi)}, && (49)\\
q_t &= (1+i_t-\tau)+\eta q_t\left[i_t(1-\phi)-r_t\right], && (50)\\
(1-\xi)\beta b_{t+1} &= b_t\left\{1-\eta\left[i_t(1-\phi)-r_t\right]\right\}, && (51)\\
1+\rho_t &= 1+i_t, && (52)
\end{aligned}
$$

where the value of $\overline{q}_{t+1}$ is given by the Value-at-Risk regulation (16) and (48) when $b_{t+1}=0$:

$$\overline{q}_{t+1}=\frac{1}{\eta}\frac{D_{t+1}}{N_{t+1}}. \tag{53}$$

Equation (49) shows that increasing by one unit today's net worth gives the average of the discounted marginal value of net worth. The banking bubble growth rate equation (51) exists only when agents believe in the bubble's existence, implying that $b_t>0$.

Appendix G shows that the transversality condition on banking stocks (4) can be written as

$$\left\{1-\eta\left[i_t(1-\phi)-r\right]\right\}/\beta(1-\xi)<1/\beta. \tag{54}$$

As shown in Chevallier and El Joueidi (2019), a banking bubble exists only under VaR regulations because it ties lending to market values. In contrast, under a regulation based on book values as in Basel I, the bubble growth is given by $b_{t+1}/b_t=1/\beta(1-\xi)$, which is ruled out by the TVC. Thus, a banking bubble cannot exist.

By tying lending to market values, LTV and VaR regulations allow the two different types of bubbles studied here to emerge according to the following mechanism. By tying lending to market values, these regulations allow positive expectations, respectively on house and bank stock prices to affect loans and, as a consequence, other macroeconomic variables. These regulations reduce the growth rate of, respectively, stock and house prices, leading transversality conditions for bank stocks and housing to be satisfied, even in the presence of stock and housing bubbles.

4.1 General equilibrium

This section defines and analyses an equilibrium in which housing bubbles and banking bubbles arise, i.e. $p_{h,t} > 0$ and $b_t > 0$.

Definition 9. The equilibrium in which both housing and banking bubbles exist simultaneously is called here the *double bubbly equilibrium.*

Denote x_t^{hb} the variables defining the equilibrium before the banking bubble bursts and $\overline{x}_t^{hb}$ the variables when the banking bubble bursts, at $t = T$. Once the banking bubble bursts, the housing bubble does not survive and the economy falls back to the bubbleless economy.[13] Variables are then denoted x_t^*. At equilibrium, the quantity of stock and total housing supply are normalized to one ($s_{t+1} = 1,\ h_{t+1} = 1\,\forall t$).

If a housing and a banking bubble simultaneously exist in t such that b_t^{hb}, $p_{h,t}^{hb} > 0$, until the bubble bursts in T, a *competitive double bubbly general equilibrium* is defined as

$$\begin{aligned}\mathcal{E}_t^{hb} \quad = \quad & \{d_t^{hb}, N_{t+1}^{hb}, K_{t+1}^{hb}, L_{t+1}^{hb}, D_{t+1}^{hb}, h_{t+1}^{hb}, M_{t+1}^{hb}, \pi_t^{hb}, y_t^{hb}, c_t^{hb}, \\ & b_t^{hb}, s_{t+1,}^{hb} q_t^{hb}, \overline{q}_t^{hb}, r_t^{hb}, i_t^{hb}, p_t^{hb}, p_t^{hb}, c_{h,t}^{hb}, w_t^{hb}, l_t^{hb}, T_t^{hb}, \rho_t^{hb}\}\ \forall t < T,\end{aligned}$$

such that taking prices as given, all agents maximise their future payoffs subject to their constraints. Transversality conditions (32) and (4) being satisfied. Also, the market for loans, for deposits, for labor, for housing, and for stocks clear, and the government's spending, σM_t^{hb}, is equal to its revenue T_t^{hb}. At $t = T$, the banking bubble bursts such that $b_t^{hb} = p_{h,t}^{hb} = 0\ \forall\, t \geq T$, and a *competitive double bubbly general equilibrium* $\overline{\mathcal{E}}_t^{hb}$ is defined as $\mathcal{E}_t^*\ \forall t \geq T$. In that equilibrium, $\overline{N}_T^{hb} = N_T^{hb}$ and, taking prices as given, all agents maximise their future expected payoffs subject to their constraints. Transversality conditions (32) and (4) being satisfied. The market for loans, for deposits, for labor, for housing, and for stocks are in equilibrium. As in the *bubbleless equilibrium*, equation (36) where superscripts * are replaced by ^{hb}, gives the condition on the goods market.

[13] On the banking bubble bursts in T, the only compatible equilibrium is the *bubbleless equilibrium.* Hence our model implies that once the banking bubble bursts, expectations on house prices are affected as well such that $p_{h,t} = 0$ for all $t \geqslant T$.

Proposition 10. *A bubbleless equilibrium and a double bubbly equilibrium can coexist if and only if* $(\xi/\eta + r)/(1-\phi) < i_t < r + \sigma$, $\forall t$.

The right hand side condition is the banking TVC (54) requirement while the left hand side is the housing TVC (33) together with the interest rate equality condition (52). The housing TVC imposes an upper bound to the lending rate and the banking stock TVC requires a lower bound.

Semi-stationary double bubbly equilibrium

As in Chevallier and El Joueidi (2019) and Weil (1987), we consider the equilibrium in which both banking and housing bubbles arise, with the following properties. The equilibrium is constant until the banking bubble collapses at $t = T$. It is defined by $\mathcal{E}_0^{hb} = ... = \mathcal{E}_{T-1}^{hb} = \mathcal{E}^{hb}$, with $b_0^{hb} = ... = b_{T-1}^{hb} = b^{hb} > 0$ and $p_{h,0}^{hb} = ... = p_{h,T-1}^{hb} = p_h^{hb} > 0$. Call this equilibrium the *semi-stationary double bubbly equilibrium.* At $t \geq T$, the banking bubble bursts, i.e. $b_t^{hb} = 0$, and the equilibrium converges to the *stationary bubbleless equilibrium* $\mathcal{E}^*$ defined in Subsection 3.1 .

The *semi-stationary double bubbly equilibrium* is characterised by the following formulas. The deposit rate is given by (2) $r = 1/\beta - 1$. Using (2) and (51), the semi-stationary lending rate is

$$i^{hb} = \frac{r(\beta + \eta) + \beta\xi}{(1-\phi)\,\eta}. \tag{55}$$

As in the *stationary bubbleless equilibrium*, the lending rate on corporate loans in the *semi-stationary double bubbly equilibrium* , i^{hb}, increases with capital regulation stringency: ϕ, the fraction of reserves, and $1/\eta$, the tightness of the Value-at-Risk regulation. In addition here, the lending rate rises with the probability ξ of burst of the banking bubble, to account for the risk of bursts.

The semi-stationary marginal values of net worth before and after the banking bubble collapses are given by (49) and (50):

$$\overline{q}^{hb} = \frac{\tau - i^{hb}}{\beta\xi}, \tag{56}$$

and

$$q^{hb} = \frac{1 - \tau + i^{hb}}{\beta(1-\xi)}. \tag{57}$$

From (53), the semi-stationary leverage ratio is

$$\frac{D^{hb}}{N^{hb}} = \eta \overline{q}^{hb}. \tag{58}$$

From the LTV regulation (7), $M^{hb} = m p_h^{hb}$, where $p_h^{hb} > 0$. The formulas for the semi-stationary interest rate, i and ρ, corporate loans, L, wage, w, labor, l, the impatient household consumption, c_h, the patient household consumption, c, and welfare, W, are the ones in Subsection 4, where superscripts ^{h} are replaced by superscripts ^{hb}. Moreover, combining (11), (12), (58),

$$N^{hb} = \frac{L^{hb} + M^{hb}}{1 + (1-\phi)\,\eta \overline{q}^{hb}}. \tag{59}$$

It can be shown that for $\overline{q}^{hb} > 0$, N^{hb} is strictly positive. Having the variable $\overline{q}^{hb}$ strictly positive is equivalent to $\eta > (\xi + r) / [r - \tau(1-\phi)]$. In the remainder of the paper, we consider that this last condition always holds.[14] This condition also implies that

$$i^{hb} < i^{*}. \tag{60}$$

Hence, $L^{hb} > L^{*}$ by the formula of L in (39) with the equilibria respective superscripts. In addition, from (58), $D^{hb} = \eta \overline{q}^{hb} N^{hb}$.

From the VaR regulation (16) and the value function when the bubble exists (48),[15] we have

$$b^{hb} = \frac{D^{hb}}{\eta} - q^{hb} N^{hb}. \tag{61}$$

4.2 Equilibria comparison

This section compares the *bubbleless* and the *double bubbly equilibria*. In particular, it analyses amplification mechanisms of having banking bubbles in addition

[14]See **Appendix I** for all the parameter conditions of the model.

[15]Note that only positive banking bubbles are considered, which is equivalent to imposing the following restriction on parameters: $\eta > [1 - \beta(1-\xi)] / [(\tau - \xi)(1-\phi) - r]$ (see equation (61)).

to a housing bubble.

Proposition 11. *The interaction of housing and banking bubbles amplifies the size of banks' balance sheet.*

As shown in Proposition 7 in the Baseline model, the housing bubble increases banks' balance sheet size. When, in addition, there is a banking bubble, the impact on banks' size is amplified. This can be seen by rewriting, using (56), (58) and (61), the banking bubble term b^{hb} as a function of the housing bubble term p_h^{hb},

$$\begin{aligned} b^{hb} &= \left(\overline{q}^{hb} - q^{hb}\right) N^{hb} \\ &= \left[\frac{\eta\left(\tau-\xi\right)\left(1-\phi\right) - r\left(\eta+\beta\right) + \beta\xi}{\beta\xi(1-\xi)\left(1-\phi\right)\eta}\right]\left[\frac{L^{hb} + mp_h^{hb}}{1+\left(1-\phi\right)\eta\overline{q}^{hb}}\right]. \end{aligned} \tag{62}$$

From equation (62), it is clear that the housing bubble inflates the banking bubble. If there are no housing bubbles such that $p_h^{hb} = 0$, the banking bubble value is smaller. The banking bubble allows banks to relax the VaR capital requirement constraint. Banks can therefore demand more deposits, raising their leverage, and grant more loans. The housing bubble increases the value of the banking bubble b^{hb}, relaxing further the VaR capital requirement constraint. Thus, the amount of loans granted is further inflated, as compared to a world with no housing bubble. The intuition is that the LTV regulation ties the housing bubble value to lending. Hence, mortgage lending is larger in a world in which agents believe in the existence of housing bubbles, and thus increase banks' size, compared to a state in which there are no housing bubbles. Similarly, when there is a housing bubble in t such that $p_{h,t} > 0$, the amount of loans granted when there are no banking bubble is fewer than in a world in which banking bubbles exist.

Proposition 12. *The stationary double bubbly equilibrium provides a larger welfare than the stationary bubbleless equilibrium.*

The proof of Proposition 12 is presented in **Appendix H.** As stated in Proposition 11 above, by relaxing the VaR capital requirement constraint, banking and housing bubbles allow banks to grant more corporate loans. First, it allows to reduce interest rates (equation (60)) as compared to the *stationary bubbleless equilibrium*. Hence, mortgage cost is reduced and labor earnings rise (see the formulas for w and l in Subsection 3.1), increasing the impatient household consumption. Second, by increasing the equilibrium quantity of corporate loans, bank's size increases as well and thus the patient household (the owner of the bank) consumption.

4.3 Equilibrium dynamics

This last section studies the local dynamics around the *stationary bubbleless equilibrium* and the *semi-stationary double bubbly equilibrium*. As in the Baseline model, we analyse the stability of the system through a numerical example. We borrow the calibration of the Baseline model except for two parameters. The cost of bank net worth and the mortgage interest deduction rate are calibrated to $\tau = 0.15$ and $\sigma = 1.5\%$. As shown in Proposition 10, under this calibration, the *bubbleless* and the *double bubbly equilibria* coexist until the banking bubble bursts.[16] Furthermore, set the value for the probability of burst of the banking bubble to $\xi = 0.1$. All the parameter restrictions are in **Appendix I**.

[16] The parameter τ has to be large enough for the banking bubble to be positive, cf. equation (96) of **Appendix I**. The parameter σ is derived from the steady state formulas (34), (38) and (23) and thus a function of the interest rate on loans. The interest rate on loans when a banking bubble exists is different from the one in the Baseline model. Thus, τ changes as well.

Table 2: Bubbleless and double bubbly equilibria steady states

	No bubble	Banking and housing bubble
	$(p_{h,t}, b_t = 0)$	$(p_{h,t}, b_t > 0)$
Variables	values	values
N	0.0473	0.0363
D	0.9085	0.9349
L	0.9467	0.9569
M	0	0.0049
$L + M$	0.9468	0.9619
p	0.0473	0.0487
b	0	0.0139
p_h	0	0.0045
i	0.0177	0.0159
c	0.0097	0.0099
c_h	1.9562	1.9740
welfare	1.9659	1.9839

Table 2 is coherent with the results summarised in Subsections 4.1 and 4.2. Compared to the *stationary bubbleless equilibrium*, the quantity of loans supplied, $L + M$, by the bank is larger in the *semi-stationary double bubbly equilibrium*. As shown above, in Subsection 4.1, by increasing the quantity of loans supplied, bubbles reduce the lending rate. Finally, welfare W in the *semi-stationary double bubbly equilibrium* is larger than the one in the *stationary bubbleless equilibrium*.

When agents do not believe bubbles exist, $b_t = p_{h,t} = 0$ for all t, as well as when agents believe in the existence of bubbles, b_t, $p_{h,t} > 0$ for $t = 0, ...T$, until the banking bubble bursts, the examination of the set of eigenvalues associated with the linearised system around the *stationary bubbleless* and the *semi-stationary double bubbly equilibria*, show that the number of unstable eigenvalues (eigenvalues that lie outside the unit circle) is equal to the number of forward looking variables. The set of eigenvalues is reported in **Appendix J**. Thus, one can observe that, under this calibration, the system of equations when $b_t = p_{h,t} = 0$ for all t and when b_t, $p_{h,t} > 0$ for all $t < T$, is determined and both the *bubbleless* and the *double bubbly equilibria* are stable and unique.

5 Conclusion

In this paper, we develop a dynamic general equilibrium model in infinite horizon, with an endogenous banking sector and two market sensitive regulatory constraints, LTV and VaR regulations. We show that when agents face an LTV regulation, two different equilibria may emerge and coexist: a *bubbleless* and a *housing bubble equilibria*. The LTV regulation permits housing bubbles to exist. In contrast, without this constraint, housing bubbles are ruled out by the TVC. The intuition is that the LTV, by tying lending to the house value, reduces the growth rate of house prices, allowing an equilibrium with non-explosive housing bubbles to emerge. Moreover, we show that the mortgage interest deduction should be large enough to allow housing bubbles to emerge. Our results also suggest that, when agents value housing, modelled via the wish to take the maximum amount of mortgage loans, housing bubbles increase banks' credit. We also show that, as long as banks face positive operational costs, housing bubbles decrease welfare. In the extension of the model, we show in addition that market sensitive macroprudential regulations, represented by LTV and VaR regulations, allow housing and banking bubbles to arise simultaneously. In particular, the existence of banking bubbles amplifies the increase in banks' credit triggered by the existence of housing bubbles. When both bubbles exist, the welfare impact is positive.

The main policy implication of this paper suggests that, by reducing the growth rate of prices, LTV constraints allow the existence of an equilibrium where housing bubbles survive and affect negatively welfare. This might explain the existence of potential housing bubbles in countries where the LTV regulation has already been introduced. It is also consistent with several research that failed to show the impact of LTV on reducing price volatility (Cerutti et al. (2017), Kuttner and Shim (2016), and Vandenbussche et al. (2015)) and research that shows that price dynamics are exacerbated by regulations that are tied to market values (Danıelsson et al., 2004).

Extensions of the model could involve studying the impact of other macroprudential tools to reduce the volatility of housing prices. Among others, the introduction of a debt-to-income together with an LTV may prevent housing bubbles from arising while reducing household and bank exposure to the real estate market risks. Furthermore, a central bank could be added in order to study the impact of conventional and non-conventional monetary policies on the existence of bubbles. Monetary policy mix combined with LTV measures may reduce bank exposure to risk while preventing housing bubbles from appearing.

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Appendices

Appendix A

This appendix shows that, without the VaR capital requirement, each bank chooses to hold the maximum amount of deposits.
Without the VaR capital requirement, the bank maximisation problem is given by

$$V_t\left(N_t, D_t, M_t\right) = Max_{\{N_{t+1},D_{t+1},M_{t+1}\}}\left[d_t + \beta V_{t+1}(N_{t+1}, D_{t+1}, M_{t+1})\right],$$

subject to

$$d_t = \left(1+i_t\right)N_t + D_t\left[i_t(1-\phi) - r_t\right] + M_t\left[(1+\rho_t) - (1+i_t)\right] - \tau N_t - N_{t+1},$$

$$N_t, D_t, M_t \geq 0 \quad \text{for all } t.$$

The problem described above can be written as,

$$\begin{aligned} V_t(N_t, D_t, M_t) = & \\ & Max_{\{N_{t+1},D_{t+1},M_{t+1}\}}\{\left(1+i_t\right)N_t + D_t\left[i_t(1-\phi) - r_t\right] \\ & + M_t\left[(1+\rho_t) - (1+i_t)\right] - \tau N_t - N_{t+1} \\ & + \beta V_{t+1}\left(N_{t+1}, D_{t+1}, M_{t+1}\right)\}. \end{aligned} \tag{63}$$

The marginal value from an increase in net worth, deposits and mortgage loans are given by

$$\frac{dV_t\left(N_t, D_t, M_t\right)}{dN_{t+1}} = -1 + \beta\frac{dV_{t+1}\left(N_{t+1}, D_{t+1}, M_{t+1}\right)}{dN_{t+1}}, \tag{64}$$

$$\frac{dV_t\left(N_t, D_t, M_t\right)}{dD_{t+1}} = \beta\frac{dV_{t+1}\left(N_{t+1}, D_{t+1}, M_{t+1}\right)}{dD_{t+1}},$$

and

$$\frac{dV_t\left(N_t, D_t, M_t\right)}{dM_{t+1}} = \beta\frac{dV_{t+1}\left(N_{t+1}, D_{t+1}, M_{t+1}\right)}{dM_{t+1}}, \tag{65}$$

Using the envelop theorem,

$$\frac{dV_t\left(N_t, D_t, M_t\right)}{dN_t} = 1 + i_t - \tau,$$

$$\frac{dV_t\left(N_t, D_t, M_t\right)}{dD_t} = i_t\left(1 - \phi\right) - r_t,$$

and

$$\frac{dV_t\left(N_t, D_t, M_t\right)}{dM_t} = \left(1 + \rho_t\right) - \left(1 + i_t\right).$$

Banks decide to hold an infinite amount of deposits if $dV_t\left(N_t, D_t, M_t\right)/dD_{t+1} > 0$, which is equivalent to

$$i_t\left(1 - \phi\right) - r_t > 0. \tag{66}$$

The interior solution for the net worth is given by $dV_t\left(N_t, D_t, M_t\right)/dN_{t+1} = 0$. Equation (64) becomes

$$i_t = \frac{1}{\beta} - 1 + \tau. \tag{67}$$

The interior solution for mortgage loans is given by the equality below: $dV_t\left(N_t, D_t, M_t\right)/dM_{t+1} = 0$. Equation (65) becomes

$$i_t = \rho_t. \tag{68}$$

From the household problem optimal condition (2) and putting (67) in (66), we get the following condition

$$\tau\beta\left(1 - \phi\right) > \phi\left(1 - \beta\right). \tag{69}$$

When condition (69) above holds, banks always choose the maximum amount of deposits. Hence, the VaR capital requirement regulation always binds. Note that the constraint $D_{t+1} > 0$ is always verified since the marginal value from deposits is positive $\left(dV_t\left(N_t, D_t, M_t\right)/dD_{t+1} > 0\right)$ and the problem is linear.

Appendix B

This appendix shows the solution of each bank maximisation problem.

Each bank problem is defined by

$$V_t\left(N_t, M_t\right) = Max_{\{N_{t+1}, M_{t+1}\}}\left\{d_t + \beta V_{t+1}\left(N_{t+1}, M_{t+1}\right)\right\},$$

subject to

$$d_t = \left(1 + i_t\right) N_t + D_t\left[i_t(1-\phi) - r_t\right] + M_t\left[\left(1+\rho_t\right) - \left(1+i_t\right)\right] - \tau N_t - N_{t+1},$$

$$D_t = \eta V_t\left(N_t, M_t\right),$$

$$N_t, D_t, M_t \geq 0 \quad \text{for all } t.$$

The Bellman equation becomes

$$\begin{aligned} V_t\left(N_t, M_t\right) =& Max_{\{N_{t+1}, M_{t+1}\}}\left(1 + i_t - \tau\right) N_t + M_t\left(\rho_t - i_t\right) \\ &+ \eta V_t\left(N_t, M_t\right)\left[i_t\left(1-\phi\right) - r_t\right] - N_{t+1} + \beta V_{t+1}\left(N_{t+1}, M_{t+1}\right). \end{aligned}$$

The marginal value from mortgage loan increase is given by

$$\frac{dV_t\left(N_t, M_t\right)}{dM_{t+1}} = \beta\frac{dV_{t+1}\left(N_{t+1}, M_{t+1}\right)}{dM_{t+1}}.$$

By the envelop theorem,

$$\frac{dV_t\left(N_t, M_t\right)}{dM_t} = \rho_t - i_t + \eta\frac{dV_t\left(N_t, M_t\right)}{dM_t}\left[i_t\left(1-\phi\right) - r_t\right].$$

The interior solution for mortgage loan is given by $dV_t\left(N_t, M_t\right)/dM_{t+1} = 0$. Therefore,

$$\frac{dV_{t+1}\left(N_{t+1}, M_{t+1}\right)}{dM_{t+1}} = 0. \tag{70}$$

The marginal value from a net worth increase is

$$\frac{dV_t\left(N_t, M_t\right)}{dN_{t+1}} = -1 + \beta\frac{dV_{t+1}\left(N_{t+1}, M_{t+1}\right)}{dN_{t+1}}.$$

By the envelop theorem,

$$\frac{dV_t\left(N_t, M_t\right)}{dN_t} = \left(1 + i_t - \tau\right) + \eta \frac{dV_t\left(N_t, M_t\right)}{dN_t}\left[i_t\left(1 - \phi\right) - r_t\right]. \tag{71}$$

The interior solution for net worth is defined by $dV_t\left(N_t, M_t\right)/dN_{t+1} = 0$. Hence,

$$\frac{dV_{t+1}\left(N_{t+1}, M_{t+1}\right)}{dN_{t+1}} = \frac{1}{\beta}.$$

Call the marginal value of net worth q_t and the marginal value of mortgage loans q_t^M. Since the problem is linear in N and M, the value function is

$$V_t\left(N_t, M_t\right) = q_t N_t + q_t^M M_t. \tag{72}$$

Furthermore, by (70), $q_t^M = 0$ for all t, and $V_t\left(N_t, M_t\right) = V_t\left(N_t\right)$.

Replacing (72) in the bank maximisation problem, the solution is given by the following system

$$\begin{aligned} q_{t+1} &= \frac{1}{\beta}, \\ q_t &= \left(1 + i_t - \tau\right) + \eta q_t\left[i_t\left(1 - \phi\right) - r_t\right], \\ i_t &= \rho_t. \end{aligned}$$

Appendix C

In this appendix, we show that without the Loan-to-Value constraint, impatient households decide to borrow a maximum amount of mortgage loans.

Define the Lagrangian of the impatient household problem without the Loan-to-Value constraint as

$$\begin{aligned} \mathscr{L}_t\left(M_{t+1}, h_{t+1}, l_t, c_t, \lambda_t\right) &= \sum_{t=0}^{\infty} \beta^t \left[c_{h,t} + n\frac{l_t^{1+\varphi}}{1+\varphi}\right] \\ &+ \sum_{t=0}^{\infty} \beta^t \lambda_t \left[M_{t+1} - \left[1 + \rho_t - \sigma\right] M_t - p_{h,t}(h_{t+1} - h_t) + w_t l_t - c_{h,t}\right], \end{aligned}$$

where λ_t is the shadow price of the household wealth.

The first order conditions are given by

$$d\mathscr{L}_t\left(M_{t+1},h_{t+1},l_t,c_t,\lambda_t\right)/dM_{t+1}=\lambda_t\beta^t-\lambda_{t+1}\beta^{t+1}\left(1+\rho_{t+1}-\sigma\right), \tag{73}$$

$$d\mathscr{L}_t\left(M_{t+1},h_{t+1},l_t,c_t,\lambda_t\right)/dc_t=\beta^t-\lambda_t\beta^t, \tag{74}$$

$$d\mathscr{L}_t\left(M_{t+1},h_{t+1},l_t,c_t,\lambda_t\right)/dl_{t+1}=\beta^t n l_t^{\varphi}+\beta^t\lambda_t w_t, \tag{75}$$

$$d\mathscr{L}_t\left(M_{t+1},h_{t+1},l_t,c_t,\lambda_t\right)/dh_{t+1}=-\beta^t\lambda_t p_{h,t}+\lambda_{t+1}\beta^{t+1}p_{h,t+1}, \tag{76}$$

$$\begin{aligned} & d\mathscr{L}_t\left(M_{t+1},h_{t+1},l_t,c_t,\lambda_t\right)/d\lambda_t= \\ & \beta^t\left\{M_{t+1}-\left[1+\rho_t-\sigma\right]M_t-p_{h,t}(h_{t+1}-h_t)+w_t l_t-c_{h,t}\right\}. \end{aligned} \tag{77}$$

Hence, impatient households hold the maximum amount of mortgage loans M_{t+1} and hence the LTV always binds if $d\mathscr{L}_t\left(M_{t+1},h_{t+1},l_t,c_t,\lambda_t\right)/dM_{t+1}>0$, which is equivalent to

$$\frac{1}{\beta}>1+\rho_{t+1}-\sigma. \tag{78}$$

By (22), (22), and (23),

$$\rho=\frac{r\left(\eta+\beta\right)+\beta\tau}{\beta+\eta\left(1-\phi\right)}. \tag{79}$$

Hence, condition (78) becomes

$$r>\frac{r\left(\eta+\beta\right)+\beta\tau}{\beta+\eta\left(1-\phi\right)}-\sigma, \tag{80}$$

which is equivalent to

$$\sigma > \frac{r\eta\phi + \beta\tau}{\beta + \eta(1-\phi)} \tag{81}$$

Moreover, the interior solution for consumption, labor supply, demand of houses, and the shadow value of wealth at time t, are given by $d\mathscr{L}_t\left(M_{t+1}, h_{t+1}, l_t, c_t, \lambda_t\right)/dc_t$, $d\mathscr{L}_t\left(M_{t+1}, h_{t+1}, l_t, c_t, \lambda_t\right)/dl_t = 0$, $d\mathscr{L}_t\left(M_{t+1}, h_{t+1}, l_t, c_t, \lambda_t\right)/dh_{t+1} = 0$, and

$d\mathscr{L}_t\left(M_{t+1}, h_{t+1}, l_t, c_t, \lambda_t\right)/d\lambda_t = 0$. Equations (74), (75), (76), and (77) then become

$$\lambda_t = -1, \tag{82}$$

$$nl_t^{\varphi} = w_t \tag{83}$$

$$p_{h,t} = \beta p_{h,t+1}, \tag{84}$$

$$c_{h,t} = M_{t+1} - \left[1 + \rho_t - \sigma\right] M_t - p_{h,t}(h_{t+1} - h_t) + w_t l_t. \tag{85}$$

Using (78) and (84) we get $p_{h,t+1}/p_{h,t} > 1 + \rho_{t+1} - \sigma$. This condition states that if the growth rate of house prices is larger than the marginal cost of mortgage loans, impatient households always borrow the maximum amount of loans. Hence, the LTV constraint always binds.

Appendix D

The housing transversality condition holds if house price growth is smaller than the household discount rate, i.e. $p_{h,t+1}/p_{h,t} < 1/\beta$. Using (34), the latter is equivalent to

$$\frac{1}{\beta + m\left[1 - \beta\left(1 + \rho_{t+1} - \sigma\right)\right]} < \frac{1}{\beta}. \tag{86}$$

Therefore, the housing transversality condition holds if

$$1 + \rho_{t+1} - \sigma < 1/\beta. \tag{87}$$

Appendix E1

This appendix shows that the welfare in a *stationary housing bubble equilibrium* is smaller than in the *stationary bubbleless equilibrium*.

From Section 3.1, $c^h = D^h (1/\beta - 1) + d^h - T^h$ and $c^* = D^* (1/\beta - 1) + d^*$. Again from Section 3.1, $d^h = \left(i^h - \tau\right) N^h + D^h \left[i^h(1-\phi) - 1/\beta + 1\right]$, $d^* = (i^* - \tau) N^* + D^* [i^*(1-\phi) - 1/\beta + 1]$ and from the government balanced budget equality, $T^h = -\sigma M^h$ Hence,

$c_h^h + c^h = D^h (1/\beta - 1) + d^h - T^h + \left[\sigma - \rho^h\right] M^h + w^h l^h$ and $c_h^* + c^* = D^* (1/\beta - 1) + d^* + w^* l^*$.

We know from Section 3.1 that $w^* l^* = w^h l^h$. We also know from (23) that $i_t = \rho_t$ for all t. Moreover, fom (46), $D = N\eta/\beta$, and from (42), i.e. $L^* = L^h$, Hence:

$c^h = c^* + M^h \left\{i^h [1 + (1-\phi)(\eta/\beta)] - \tau\right\} / [1 + (1-\phi)(\eta/\beta)] - \sigma M^h = c^* - M^h \left(\sigma - i^h + \tau / [1 + (1-\phi)(\eta/\beta)]\right)$, and

$c_h^h = c_h^* + M :^h \left(\sigma - i^h\right)$.

Hence $c^h + c_h^h = c^* + c_h^* - M^h \tau / [1 + (1-\phi)(\eta/\beta)]$.

Therefore, $W^h < W^*$if τ>0 which is always satisfied.

Appendix E

This appendix shows that the welfare in a *stationary housing bubble equilibrium* is smaller than in the *stationary bubbleless equilibrium*.

From Section 3.1, $c^h = D^h (1/\beta - 1) + d^h - T^h$ and $c^* = D^* (1/\beta - 1) + d^*$. Again from Section 3.1, $d^h = \left(i^h - \tau\right) N^h + D^h \left[i^h(1-\phi) - 1/\beta + 1\right]$, $d^* = (i^* - \tau) N^* + D^* [i^*(1-\phi) - 1/\beta + 1]$ and from the government balanced budget equality, $T^h = -\sigma M^h$ Hence,

$c_h^h + c^h = D^h (1/\beta - 1) + d^h - T^h + \left[\sigma - \rho^h\right] M^h + w^h l^h$ and $c_h^* + c^* = D^* (1/\beta - 1) + d^* + w^* l^*$. Hence, by (40) and (44),

$$W^h > W^*$$

if

$$\left(i^h-\tau\right)N^h+D^h\left[i^h(1-\phi)\right]-\sigma M^h+\left[\sigma-\rho^h\right]M^h+w^hl^h>\left(i^*-\tau\right)N^*+D^*\left[i^*(1-\phi)\right]+w^*l^*. \tag{88}$$

We know from Section 3.1 that $w^*l^*=w^hl^h$. We also know from (23) that $i_t=\rho_t$ for all t. Hence, $W^h>W^*$ if

$$\left(i^h-\tau\right)N^h+D^h\left[i^h(1-\phi)\right]-i^hM^h>\left(i^*-\tau\right)N^*+D^*\left[i^*(1-\phi)\right],$$

which is equivalent to

$$\left(i^*(1-\phi)\right)\left(D^h-D^*\right)+\left(i^*-\tau\right)\left(N^h-N^*\right)>i^*M^h. \tag{89}$$

From (46), $D=N\eta/\beta$. Therefore, (89) can be simplified into

$$i^*(1-\phi)\frac{\eta}{\beta}(N^h-N^*)+\left(i^*-\tau\right)\left(N^h-N^*\right)>i^*M^h,$$

and

$$\left[i^*(1-\phi)\frac{\eta}{\beta}+\left(i^*-\tau\right)\right]\left(N^h-N^*\right)>i^*M^h.$$

From (42), i.e. $L^*=L^h$, and (45), $N^h=\left[L^h+M^h\right]/\left[1+(1-\phi)\eta 1/\beta\right]=N^*+M^h/\left[1+(1-\phi)(\eta/\beta)\right]$. Thus, the above becomes

$$\left[i^*(1-\phi)\frac{\eta}{\beta}+\left(i^*-\tau\right)\right]\left(\frac{M^h}{1+(1-\phi)(\eta/\beta)}\right)>i^*M^h,$$

which can be simplified into

$$\left[i^*(1-\phi)\frac{\eta}{\beta}+\left(i^*-\tau\right)\right]>i^*\left[1+(1-\phi)(\eta/\beta)\right],$$

and

$$0>\tau.$$

The above is never satisfied since $\tau > 0$. Hence,

$$W^h < W^*. \tag{90}$$

Appendix F

Table 3 below displays the eigenvalues associated with the linearised system around the *stationary bubbleless* and the *stationary housing bubble equilibria*.

Table 3: Eigenvalues of the bubbleless and housing bubble equilibria

no bubble ($p_{h,t} = 0$)	housing bubble ($p_{h,} > 0$)
values	values
1.15E-30	1.15E-30
1.26E-22	6.68E-20
1.52E-19	1.52E-19
1.20E-17	1.20E-17
0.95	0.95
1	1
1.01	1.01
3.16E+16	3.16E+16
2.69E+17	2.69E+17
6.41E+18	6.41E+18
Inf	Inf
Inf	Inf

Appendix G

This appendix presents the conditions that ensure that the the banking stock transversality condition (4) is satisfied. The following transversality condition is required

$$lim_{t\to\infty}\beta^t p_t = lim_{t\to\infty}\beta^t \left[\xi \left(\overline{q}_t N_t\right) + (1-\xi)\left(q_t N_t + b_t\right)\right] = 0.$$

It is satisfied if

$$lim_{t\to\infty}\beta^t \left[\xi \overline{q}_t N_t + (1-\xi)\, q_t N_t\right] = lim_{t\to\infty}(1-\xi)\, b_t \beta^t = 0.$$

Using equation (49), the transversality condition can be rewritten as follows

$$lim_{t\to\infty}\beta^t q_t N_t = lim_{t\to\infty}\left(1-\xi\right) b_t\beta^t = 0.$$

Since the bubble growth rate is

$$\frac{b_{t+1}}{b_t} = \frac{1}{\beta\left(1-\xi\right)}\left\{1-\eta\left[i_t\left(1-\phi\right)-r_t\right]\right\},$$

the TVC requires that

$$\frac{1}{\beta\left(1-\xi\right)}\left\{1-\eta\left[i_t\left(1-\phi\right)-r_t\right]\right\} < \frac{1}{\beta}.$$

Thus, the condition to allow a banking bubble to exist is

$$\eta\left[i_t\left(1-\phi\right)-r_t\right] > \xi.$$

Appendix H

This appendix shows the conditions under which the *double semi-stationary equilibrium* welfare is larger than the welfare in the *bubbleless stationary equilibrium*.

From (1) and (25),

$$c_t = D_t\left(1+r_t\right) + s_t\left(p_t+d_t\right) + \pi_t - D_{t+1} - s_{t+1}p_t - T_t,$$

$$c_{h,t} = M_{t+1} - \left[1+\rho_t-\sigma\right]M_t - p_{h,t}(h_{t+1}-h_t) + w_t l_t.$$

In a stationary or semi-stationary steady state, welfare $W = c + c_h$. From (18) and the government balanced budget equality, $T^h = -\sigma M^h$,

$W = Dr + d - \sigma M - \left[i-\sigma\right]M + wl$. Hence, $W = wl + iL - \tau N$.

Therefore, $W^* < W^{hb}$ if $w^*l^* + i^*L^* - \tau N^* < w^{hb}l^{hb} + i^{hb}L^{hb} - \tau N^{hb}$. Since, from (59), $N^{hb} = \left[L^{hb}+M^{hb}\right]/\left[1+\left(1-\phi\right)\eta\overline{q}^{hb}\right]$ and from (29) $p_h^{hb} = M^{hb}/m$, $W^* < W^{hb}$ if

$$f > p_h^{hb},$$

where

$$f = \frac{1}{m}\left(\frac{1+(1-\phi)\,\eta\overline{q}^{hb}}{\tau}\right)\left(-w^{*}l^{*}-\rho^{*}L^{*}+\tau N^{*}+w^{hb}l^{hb}+i^{hb}L^{hb}\right)-\frac{L^{hb}}{m}. \quad (91)$$

Now, we will show that $f > p_h^{hb}$ always holds considering that $W > 0$ (i.e. $c_{h,t}$, $c_t > 0$).

$W^{hb} > 0$ if $w^{hb}l^{hb}+i^{hb}L^{hb} > \tau N^{hb}$. From (59), $N^{hb} = \left[L^{hb}+M^{hb}\right] / \left[1+(1-\phi)\,\eta\overline{q}^{hb}\right]$ and from (29), $p_h^{hb} = M^{hb}/m$. Thus , $w^{hb}l^{hb}+i^{hb}L^{hb} > \tau N^{hb}$ is equivalent to

$$p_h^{hb} < \frac{1}{m}\left(\frac{1+(1-\phi)\,\eta\overline{q}^{hb}}{\tau}\right)\left(w^{hb}l^{hb}+\rho^{hb}L^{hb}\right)-\frac{L^{hb}}{m}. \quad (92)$$

Under (92), $f > p_h^{hb}$ always holds.

Appendix I

This appendix summarizes the restrictions on parameters imposed in the model:

The VaR regulation binds when

$$\tau\beta\,(1-\phi) > \phi\,(1-\beta)\,. \quad (93)$$

In the model without banking bubble, the LTV always binds and the housing TVC is always satisfied if

$$\sigma > \frac{r\eta\phi+\beta\tau}{\beta+\eta\,(1-\phi)}. \quad (94)$$

In the model with a banking bubble, we have the two following additional conditions:

$$\eta > (\xi+r)\,/\,[r-\tau(1-\phi)]\,, \quad (95)$$

which is the parameter restriction that makes $\overline{q}^{hb} > 0$, which ensures that N^{hb} is strictly positive, and

$$\eta > [1-\beta(1-\xi)]\,/\,[(\tau-\xi)(1-\phi)-r]\,. \quad (96)$$

Satisfying (96) ensures only positive banking bubbles are considered.

Appendix J

Table 4 displays the eigenvalues associated with the linearised system around the *stationary bubbleless* and the *semi-stationary double bubbly equilibria*.

Table 4: Eigenvalues of the bubbleless and the semi-stationary double bubbly equilibria

No bubble ($p_{h,t}, b_t = 0$)	Housing and banking bubble ($p_{h,} > 0, b_t > 0$)
values	values
8.57E-33	3.13E-19
4.68E-22	1.23E-17
1.52E-19	5.86E-17
7.62E-18	5.94E-15
0.95	0.95
1.002	1
1.01	1.01
2.28E+16	1.051
3.48E+17	4.92E+16
5.75E+18	2.51E+17
Inf	6.20E+17
Inf	2.39E+18
	1.76E+19